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THE GEODESIC LINES ON THE ANCHOR RING

A DISSERTATION

SUBMITTED TO THE FACULTIES OF THE GRADUATE SCHOOLS OF ARTS,
LITERATURE AND SCIENCE, IN CANDIDACY FOR THE
DEGREE OF DOCTOR OF PHILOSOPHY
(DEPARTMENT OF MATHEMATICS)

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THE GEODESIC LINES ON THE ANCHOR RING*

BY GILBERT AMES BLISS

MANGOLDT† has classified the points upon any surface according to the character of the geodesic lines passing through them. If every geodesic line through a given point remains a minimum throughout its entire length, the point is said to be of the first kind, otherwise of the second kind. He found some interesting results. For example, while upon surfaces for which the curvature is everywhere negative Jacobi has shown the points to be all of the first kind, upon those for which the curvature is everywhere positive, Mangoldt finds that points of the second kind always exist. In the latter case if there are points of the first kind the surface itself can not be closed, and all such points lie in a finite region. These regions have been investigated by Mangoldt‡ and Braunnmühl§ for surfaces of the second degree which have positive curvature.

The object of the present paper is the consideration of the geodesic lines upon the anchor ring, a surface for which the curvature is positive at some points and negative at others. The geometrical character of the lines will be discussed, and the points of the ring classified according to Mangoldt's scheme.

1. The Geometrical and Analytical Definitions of Conjugate Points. Two points a and a' upon a geodesic line are said to be conjugate to each other if every arc ab , where b lies between a and a' , is a minimum with respect to all the lines of its neighborhood joining a and b , and if also the arc ab ceases to be a minimum as soon as b lies beyond a' . The envelope of the geodesic lines through a touches the given line ab at the point a' , and the arc aa' is not a minimum unless this envelope has a singular point of a peculiar kind at a' .|| Weierstrass has shown¶ that when a second geodesic line

* Read before the Chicago Section of the American Mathematical Society, Dec. 27, 1900.

† Geodätische Linien auf positiv gekrümmten Flächen, *Journal für Mathematik*, vol. 91, (1881), p. 23.

‡ *l. c.*

§ Über Envelopen geodätischer Linien, *Mathematische Annalen*, vol. 14 (1879), p. 557; Geodätische Linien und Enveloppen auf drei-axigen Flächen zweiten Grades, *ibid.* vol. 20 (1882), p. 557.

|| Osgood, On the existence of a minimum of the integral $\int_{x_0}^{x_1} F(x, y, y') dx$ when x_0 and x_1 are conjugate points, *Transactions Amer. Math. Soc.*, vol. 2 (1901), p. 166. See also Kneser, *Variationsrechnung*, p. 97.

¶ Lectures on the Calculus of Variations at the University of Berlin, 1879.

revolves about a and approaches ab , its intersection with the latter approaches a' as a limit.

In the following paragraphs the analytical method employed in determining the geodesic lines and conjugate points is that of the calculus of variations as developed by Weierstrass.* The results are here briefly stated. Suppose a curve C_0 ,

$$x = \phi(u), \quad y = \psi(u),$$

and an integral,

$$I = \int_{u_0}^{u_1} F(x, y, x', y') du,$$

taken along C_0 from $a(u = u_0)$ to $b(u = u_1)$. If I is to be independent of the parametric representation of the curve, then F must satisfy the functional relation†

$$F(x, y, \kappa x', \kappa y') = \kappa F(x, y, x', y'), \quad \kappa > 0.$$

From this equation it can be shown that a function F_1 exists defined as follows :

$$F_1 = \frac{1}{y'^2} F_{x'x} = \frac{-1}{x' y'} F_{x'y'} = \frac{1}{x'^2} F_{y_1 y_1}. ‡$$

Suppose, as is the case in most examples,

$$\text{I} \quad F_1 > 0 \quad \text{along the arc } ab.$$

Then C_0 may make I a minimum, but if it does, it must satisfy also Legendre's condition,

$$\text{II} \quad G_1 = F_x - \frac{d}{du} F_{x'} = 0, \quad G_2 = F_y - \frac{d}{du} F_{y'} = 0.$$

The equations $G_1 = 0$ and $G_2 = 0$ are not independent, for it can be shown that§

$$x' G_1 + y' G_2 = 0.$$

Either equation is of the second order. Kneser|| has shown that it is possi-

* *l. c.* See also Kneser, *Variationsrechnung*; and Osgood, Sufficient conditions in the Calculus of Variations, *ANNALS OF MATHEMATICS*, ser. 2, vol. 2 (1901), p. 105.

† Kneser, *l. c.*, p. 7.

‡ Literal subscripts denote partial differentiation.

§ Kneser, *l. c.*, p. 10.

|| *l. c.*, §§ 29, 31.

ble to find a system of curves which are solutions of the equations II, and which have the form,

$$x = \phi(u, a, \beta), \quad y = \psi(u, a, \beta),$$

where a and β are arbitrary constants. Suppose that C_0 is the one of these solutions defined by $a = a_0, \beta = \beta_0$. Then the arc ab will make I a minimum if besides I and II the following condition is also satisfied :

$$\text{III} \quad \Theta(u, u_0) = \begin{vmatrix} \theta_1(u) & \theta_1(u_0) \\ \theta_2(u) & \theta_2(u_0) \end{vmatrix} \neq 0 \quad \text{for } u_0 < u \leq u_1,$$

where

$$\theta_1(u) = \begin{vmatrix} x_a & x' \\ y_a & y' \end{vmatrix}, \quad \theta_2(u) = \begin{vmatrix} x_\beta & x' \\ y_\beta & y' \end{vmatrix},$$

and in the derivatives x_a, x' , etc., $a = a_0, \beta = \beta_0$. The conjugate point a' is the point on C_0 where condition III ceases to hold, that is, where $\Theta(u, u_0)$ first becomes zero. Conditions I, II and III are sufficient conditions for a minimum, but not necessary.

If the equations of the anchor ring are in the form

$$x = x(\phi, \psi), \quad y = y(\phi, \psi), \quad z = z(\phi, \psi),$$

the integral expressing the length of the arc of any curve

$$\phi = \phi(u), \quad \psi = \psi(u)$$

will be

$$s = \int_{u_0}^{u_1} \sqrt{E\phi'^2 + 2F\phi'\psi' + G\psi'^2} du$$

where E, F , and G have the values

$$E = \left(\frac{\partial x}{\partial \phi}\right)^2 + \left(\frac{\partial y}{\partial \phi}\right)^2 + \left(\frac{\partial z}{\partial \phi}\right)^2, \quad G = \left(\frac{\partial x}{\partial \psi}\right)^2 + \left(\frac{\partial y}{\partial \psi}\right)^2 + \left(\frac{\partial z}{\partial \psi}\right)^2,$$

$$F = \frac{\partial x}{\partial \phi} \frac{\partial x}{\partial \psi} + \frac{\partial y}{\partial \phi} \frac{\partial y}{\partial \psi} + \frac{\partial z}{\partial \phi} \frac{\partial z}{\partial \psi}.$$

The geodesic lines are the curves upon the surface which make this integral a minimum. The object of the present paper is therefore threefold : the discussion of (1) the function F_1 for the integral s , (2) the curves which satisfy

$G_1 = 0$ or $G_2 = 0$ for this integral, and (3) the conjugate points defined by the zeros of $\Theta(u, u_0)$.

2. The Equations of the Geodesic Lines on the Anchor Ring.

The equations of the anchor ring can be written in either of the two following forms :

$$(1) \quad \begin{cases} x = (a + b \cos \phi) \cos \psi = r \cos \psi, \\ y = (a + b \cos \phi) \sin \psi = r \sin \psi, \\ z = b \sin \phi = \sqrt{b^2 - (r - a)^2}, \end{cases}$$

where the quantities involved have the meanings indicated in figure 1. The integral s is

$$(2) \quad s = \int_{u_0}^{u_1} F(\phi, \psi, \phi', \psi') du,$$

where

$$F(\phi, \psi, \phi', \psi') = \sqrt{b^2 \phi'^2 + (a + b \cos \phi)^2 \psi'^2}.$$

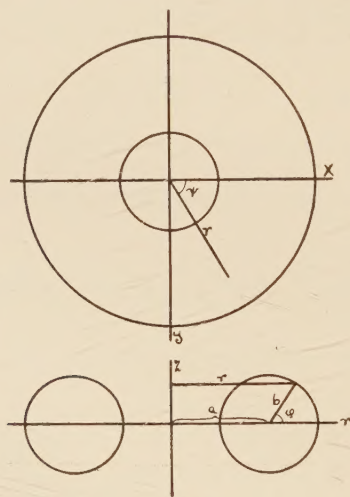


FIG. 1.

The function F_1 is found to be

$$F_1 = \frac{b^2(a + b \cos \phi)^2}{F^3},$$

which is always > 0 . Condition I is therefore satisfied. After one integration the differential equation $G_2 = 0$ becomes

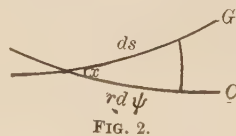
$$(3) \quad F_{\psi'} = \frac{(a + b \cos \phi)^2 \psi'}{F} = a.$$

The geodesic lines to be investigated are the solutions of this equation.

The constant of integration a has a simple geometrical meaning. It can always be taken positive, for a negative value in (3) only makes ψ decrease instead of increase with u . From the same equation

$$(4) \quad a = r^2 \frac{d\psi}{ds} = r \cos \chi,$$

where χ is the angle between the geodesic line G and the parallel circle C of radius r (figure 2). The last equation shows that a must always lie between 0 and $a + b$. The geodesic line can cut the inner equator ($r = a - b$) only if $a \leq a - b$, and if $a > a - b$ the line touches and lies without the circle $r = a$.



The equation (3) when solved for ψ' becomes

$$(5) \quad \psi'^2 = \frac{b^2 a^2 \phi'^2}{(a + b \cos \phi)^2 [(a + b \cos \phi)^2 - a^2]} = \frac{b^2 a^2 r'^2}{r^2 (r^2 - a^2) [b^2 - (r - a)^2]}.$$

The complete integral with r as the parameter is

$$(6) \quad \begin{cases} \phi = \cos^{-1} \frac{r - a}{b}, \\ \psi = \int \frac{b a dr}{r \sqrt{R(r)}} + \beta', \end{cases}$$

where

$$R(r) = (r^2 - a^2) [b^2 - (r - a)^2],$$

and β' is an arbitrary constant.

It will be convenient for the discussion of the geodesic lines to have ϕ and ψ expressed in terms of Weierstrassian functions. For this purpose introduce the new independent variable u by means of the transformation,*

$$(7) \quad r = a + b + \frac{R'(a + b)}{4 [\wp(u) - \frac{1}{24} R''(a + b)]}.$$

* Enneper *Elliptische Functionen*, 1890, p. 30.

Here $a + b$ is a root of $R(r) = 0$. If the other three roots are denoted by

$$a_\lambda = a - b, \quad a_\mu = a, \quad a_\nu = -a,$$

then equation (7) solved for $\wp(u)$ gives the following expressions for e_λ, e_μ, e_ν , and their differences;

$$(8) \begin{cases} e_\lambda = \frac{1}{24} R''(a+b) - \frac{R'(a+b)}{8b}, & e_\mu - e_\nu = ba, \\ e_\mu = \frac{1}{24} R''(a+b) - \frac{R'(a+b)}{4(a+b-a)}, & e_\nu - e_\lambda = \frac{1}{4}(a+b-a)(-a+b-a), \\ e_\nu = \frac{1}{24} R''(a+b) - \frac{R'(a+b)}{4(a+b+a)}, & e_\lambda - e_\mu = \frac{1}{4}(a+b+a)(a-b-a). \end{cases}$$

Two cases must be distinguished according to the relative values of e_λ, e_μ, e_ν ;

Case A: $0 < a < a - b$; $e_\lambda > e_\mu > e_\nu$, $\therefore \lambda = 1, \mu = 2, \nu = 3$.

Case B: $a - b < a < a + b$; $e_\mu > e_\lambda > e_\nu$, $\therefore \lambda = 2, \mu = 1, \nu = 3$.

Since

$$(9) \quad \frac{dr}{du} = \sqrt{R(r)},$$

the integral expressing ψ is transformed by the substitution (7) into the form

$$\psi = \int \frac{ba}{r} du + \beta'.$$

To find the value of this integral in terms of the Weierstrassian functions it is necessary first to express the integrand in the same manner. Two quantities v_1 and v_2 can be defined by the equations,

$$\begin{aligned} \wp(v_1) &= \frac{1}{24} R''(a+b) - \frac{R'(a+b)}{4(a+b)}, \\ \wp(v_2) &= \frac{1}{24} R''(a+b). \end{aligned}$$

Then

$$(10) \quad r = (a+b) \frac{\wp(u) - \wp(v_1)}{\wp(u) - \wp(v_2)}.$$

The function ba/r is an elliptic function whose only poles are the points where $r = 0$, that is $u = \pm v_1 + 2m\omega_1 + 2m'\omega_3$, m and m' being any integers. The residues at $u = \pm v_1$ can be found by expansion. Since the values of dr/du in (9) at $+v_1$ and $-v_1$ differ only in sign, the developments of r at

these points begin as follows :

$$\begin{aligned} &+ a \sqrt{a^2 - b^2} (u - v_1) + \dots \dots \dots, \\ &- a \sqrt{a^2 - b^2} (u + v_1) + \dots \dots \dots. \end{aligned}$$

The residues of ba/r are therefore

$$R_{v_1} = \frac{b}{\sqrt{a^2 - b^2}} = -R_{-v_1}.$$

Any elliptic function can be expressed in terms of Weierstrassian functions when its poles and their principal parts* are known. In the present case the difference

$$\frac{ba}{r} - \frac{b}{\sqrt{a^2 - b^2}} \left[\zeta(u - v_1) - \zeta(u + v_1) \right]$$

is an elliptic function with no poles, and by a well-known theorem† must therefore be a constant. The value of the constant is found by putting $u = 0$, which makes $\wp(u)$ infinite and $r = a + b$. Then

$$\frac{ba}{r} = \frac{b}{\sqrt{a^2 - b^2}} \left[\zeta(u - v_1) - \zeta(u + v_1) + 2\zeta(v_1) \right] + \frac{ba}{a + b}.$$

By a simple integration the equations (6) become

$$(11) \quad \begin{cases} \phi = \cos^{-1} \frac{r - a}{b}, \\ \psi = \int_{\beta}^u \frac{ba}{r} du = G(u) - G(\beta), \end{cases}$$

where

$$(12) \quad G(u) = \frac{b}{\sqrt{a^2 - b^2}} \left[\log \frac{\sigma(v_1 - u)}{\sigma(v_1 + u)} + 2u\zeta(v_1) \right] + \frac{ba}{a + b} u.$$

When the value of r from (10) is substituted, these equations have the form

$$\phi = \phi(u, a, \beta), \quad \psi = \psi(u, a, \beta),$$

a being involved explicitly and through the branch points of $y^2 = R(r)$ from which the $\wp$ -, ζ -, and σ -functions are determined. They also satisfy the differential equation $G_2 = 0$. Therefore equations (11) represent a doubly infinite system of geodesic lines on the anchor ring.

* Hauptteil, cf. *Encyclopædie* II, B. 1, §9.

† Harkness and Morley, *Introduction to the Theory of Analytic Functions*, 1898, p. 255.

3. The Geometrical Character of the Geodesic Lines. It is proposed to show that the geodesic lines on the anchor ring are the meridian circles, the two equators, or curves of the types shown in figures 3-5. The solutions of (3) for certain values of a are simple. For example, when $a = 0$ or $a = b$, the only solutions are the meridian circles and the outer equator. When $a = a - b$ the inner equator is a solution for which ϕ remains constant. It remains then to discuss the geodesic lines for Case *A*, Case *B*, and the lines for $a = a - b$ along which ϕ is variable.

The position of a point upon the anchor ring is completely determined as soon as r , ψ , and z are known. The expression defining z for a point upon a geodesic line is found from (1) and (10),

$$z = \pm \sqrt{2b(a+b)} [\wp(v_1) - \wp(v_2)] \frac{\sqrt{\wp(u) - e_\lambda}}{\wp(u) - \wp(v_2)},$$

or since*

$$\sqrt{\wp(u) - e_\lambda} = \frac{\sigma_\lambda(u)}{\sigma(u)},$$

$$(13) \quad z = \pm \sqrt{2b(a+b)} [\wp(v_1) - \wp(v_2)] \frac{\sigma_\lambda(u)}{\sigma(u) [\wp(u) - \wp(v_2)]}.$$

Consider now the values of r , ψ , and z in (10), (11), and (13). In the first place they represent similar curves when a is the same but β has different values, for a change in β does not alter r or z , and only adds a constant to ψ . *The form of the geodesic line depends therefore only upon the value of a . If a is fixed, different values of β define curves which can be made to coincide by rotations about the z -axis.*

From the equations†

$$\begin{aligned} \sigma_1(u + 2\omega_1) &= -e^{2\eta_1(u+\omega_1)} \sigma_1(u), & \sigma(u + 2\omega_1) &= -e^{2\eta_1(u+\omega_1)} \sigma(u), \\ \wp(u + 2\omega_1) &= \wp(u), \end{aligned}$$

it follows that for curves of Case *A*, r and z are periodic with the period $2\omega_1$. Since also

$$(14) \quad \sigma(-u) = -\sigma(u),$$

* Schwarz, *Formeln und Lehrsätze zum Gebrauche der elliptischen Funktionen*, p. 21.

† Schwarz, *l. c.*, p. 22.

equations (11) and (12) show that

$$\psi(u + 2\omega_1) = \psi(u) + 2\Psi,$$

where 2Ψ is a constant:

$$(15) \quad 2\Psi = G(2\omega_1).$$

Any geodesic line of Case A consists of repetitions of the part for which $-\omega_1 \leq u \leq \omega_1$, and two similar parts can be made to coincide by rotations about the z -axis.

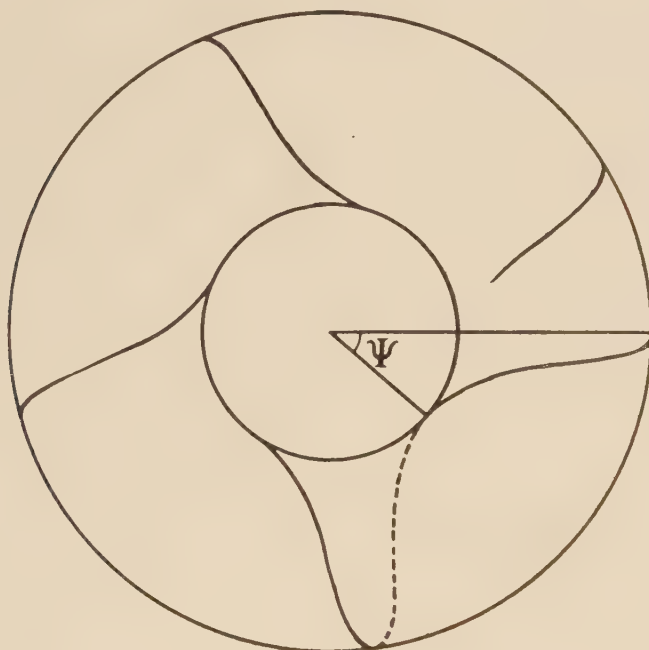


FIG. 3.

Since the form of the line depends only upon α , suppose $\beta = 0$. From (14) and the equations

$$\zeta(-u) = -\zeta(u), \quad \wp(-u) = \wp(u),$$

it follows that

$$r(-u) = r(u), \quad \psi(-u) = -\psi(u), \quad z(-u) = -z(u).$$

The curve for $-\omega_1 \leq u \leq 0$ is the reflection of the part for $0 \leq u \leq \omega_1$, first through the xz -plane and then through the xy -plane.

Since*

$$\varphi(\omega_1) = e_1,$$

r has the values $a \pm b$ for $u = 0$ and $u = \omega_1$ respectively. Between these two values of u , $\psi' = \frac{ba}{r}$ is positive, and since r must diminish near $u = 0$, the expression (9) for dr/du must be negative. *In Case A the geodesic line winds about the ring crossing the two equators alternately. The parts between any*

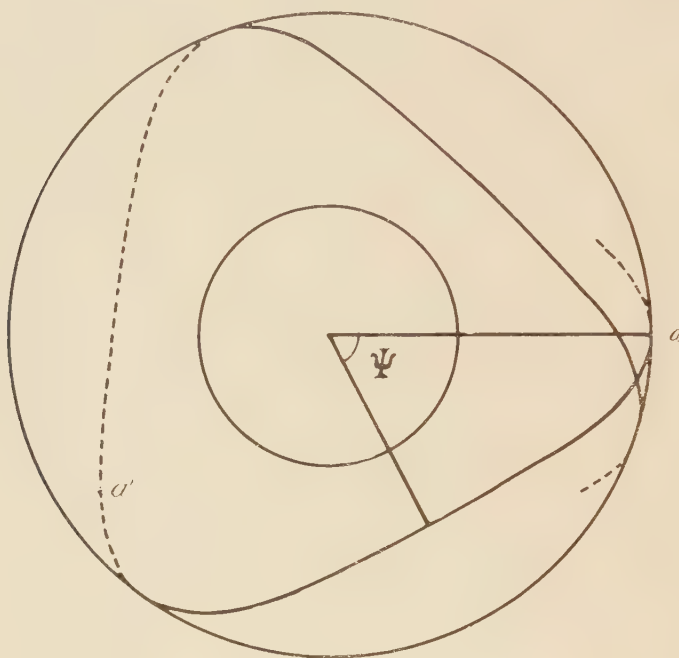


FIG. 4.

two successive intersections with either equator are similar and can be made to coincide by rotations about the z -axis (see figure 3). The two lines for the opposite values of z in (13) have the same shape, but wind about the ring in opposite directions.

The difference in the discussion for Case *B* arises because λ and μ are

* Schwarz, *l. c.*, p. 12.

interchanged. The equation

$$\sigma_2(u + 2\omega_1) = e^{2\eta_1(u + \omega_1)} \sigma_2(u)$$

shows that z has the period $4\omega_1$ instead of $2\omega_1$, and the part of the curve for $-2\omega_1 \leq u \leq 0$ is the reflection of the part for $0 \leq u \leq 2\omega_1$. In the intervals $0 \leq u \leq \omega_1$ and $\omega_1 \leq u \leq 2\omega_1$, dr/du has opposite signs, and r varies



FIG. 5.

between the values $a + b$ and a . So that in *Case B* the geodesic line winds back and forth across the outer equator and is tangent alternately to the parallel circles $r = a$, $z > 0$, and $r = a$, $z < 0$. The parts between successive contacts with either of these circles are similar, and can be made to coincide by rotations about the z -axis (see figure 4). The two curves for the opposite values of z can in this case be made to coincide.

When $a = a - b$ in (6), a curve is found for which ϕ is not a constant. The integral can be expressed by means of logarithms. Substitute

$$r = (a - b) \frac{a + b - bu^2}{a - b + bu^2}.$$

Then

$$\begin{aligned} \psi &= \sqrt{\frac{b}{a+b}} \log \sqrt{\frac{1+u}{1-u}} - \frac{b}{a+b} \log \frac{\sqrt{a+b} + u\sqrt{b}}{\sqrt{a+b} - u\sqrt{b}} + C, \\ z &= \pm 2b\sqrt{a+b} \frac{u\sqrt{1-u^2}}{a-b+bu^2}, \end{aligned}$$

where C is an arbitrary constant. These equations have properties similar to those of Case A . When $C = 0$ and u increases from 0 to 1, r decreases from $a + b$ to $a - b$, and ψ increases from 0 to ∞ . The curve for negative values of u is the reflection of the positive part. Therefore when $a = a - b$, equations (6) represent a geodesic line which cuts the outer equator in only one point. Proceeding in opposite directions from this point the curve lies on opposite sides of the xy -plane and approaches the inner equator asymptotically from above and below (see figure 5). As in Case A the lines for opposite values of z wind in opposite directions, and can not be made to coincide.

4. Relation Between the Geodesic Lines for Different Values of the Constant a . The changes which the geodesic line undergoes as a varies from 0 to $a + b$ can be found by forming the derivative Ψ_a . From (15)

$$2\Psi_a = G_a(2\omega_1) + 2G_u(2\omega_1) \frac{d\omega_1}{da}.$$

If r is expanded by Taylor's formula at the point $u = 2\omega_1$ by means of (9), and the derivatives r_a and r_u taken, it is found that

$$2 \frac{d\omega_1}{da} = - \left[\frac{r_a}{r_u} \right]_{u=2\omega_1}.$$

The value of Ψ_a is then

$$2\Psi_a = H(2\omega_1),$$

where

$$H(u) = \frac{1}{r_u} \begin{vmatrix} G_a & G_u \\ r_a & r_u \end{vmatrix}.$$

The function $H(u)$ is useful in forming $\Theta(u, u_0)$ as well as Ψ_a . It can be calculated in terms of the Weierstrassian functions from the properties of its derivative, which is expressible in terms of r . Since from (11)

$$G' = \frac{ba}{r},$$

it follows that

$$H'(u) = \frac{b}{r} \left\{ 1 - a \frac{d}{du} \frac{r_a}{r_u} \right\}.$$

Consider the functions $\frac{1}{2} \left\{ \frac{1}{r+a} - \frac{1}{r-a} \right\}$ and $\frac{d}{du} \frac{r_a}{r_u}$. The former is an elliptic function whose poles are at $u = \omega_\mu$ and $u = \omega_\nu$, where r has the values a and $-a$ respectively. The expansions can be found by means of (9). It turns out that the only negative powers are of order two, and these have the coefficients

$$(16) \quad R_\mu = \frac{-2}{R'(a)}, \quad R_\nu = \frac{+2}{R'(-a)}.$$

When $u = 0$ the function has the value $\frac{2ba}{R'(a+b)}$.

But these are exactly the properties of $\frac{d}{du} \frac{r_a}{r_u}$. From (10)

$$(17) \quad \frac{r_a}{r_u} = \frac{1}{\wp'(u)} \frac{\partial \wp(u)}{\partial a} + C_1 \frac{\wp(u)}{\wp'(u)} + \frac{C_2}{\wp'(u)},$$

where C_1 and C_2 are constants. a is involved in $\wp(u)$ through the branch points $a_\mu = a, a_\nu = -a$, so that

$$\frac{\partial \wp(u)}{\partial a} = \frac{\partial \wp(u)}{\partial a_\mu} - \frac{\partial \wp(u)}{\partial a_\nu}.$$

A formula for the derivation of $\sigma(u)$ with respect to a branch point a has been derived by Bolza:*

$$R'(a) \frac{\partial \sigma}{\partial a} = \frac{1}{12} G_2 u^2 \sigma + \frac{1}{12} R''(a) \left[u \frac{\partial \sigma}{\partial u} - \sigma \right] + \frac{\partial^2 \sigma}{\partial u^2}.$$

* The elliptic σ -functions considered as a special case of the hyperelliptic σ -functions, *Transactions Amer. Math. Soc.*, vol. 1, 1900, p. 63.

One for $\wp(u)$ can be found by dividing this by $\sigma(u)$, differentiating twice, and using the formulae*

$$\frac{\sigma''}{\sigma} = \zeta^2 - \wp, \quad \wp'' = 6\wp^2 - \frac{g_2}{2}.$$

Then

$$(18) \quad R'(a) \frac{\partial \wp}{\partial a} = \frac{1}{12} R''(a) \left[2\wp + u\wp' \right] + 2\zeta\wp' + 4\wp^2 - \frac{1}{2}g_2.$$

From (17) and (18) it is seen that $\frac{d}{du} \frac{r_a}{r_u}$ is an elliptic function with poles possible at $\omega_\lambda, \omega_\mu, \omega_\nu$, where $\wp'(u)$ vanishes. At these points r has the values $a-b, a, -a$ respectively, and can be expanded in powers of $u - \omega_\rho$ ($\rho = \lambda, \mu, \nu$) by means of (9). The quotient r_a/r_u is then found to have poles at ω_μ , and ω_ν , but none at ω_λ ; and its derivative has therefore poles at ω_μ and ω_ν , with the same principal parts R_μ, R_ν , as in (16). Similarly the derivatives of the expansion of r at $u = 0$ give $\frac{d}{du} \frac{r_a}{r_u}$ the same value $\frac{2ba}{R'(a+b)}$ when $u = 0$.

The difference between the two functions just considered is an elliptic function with no poles, and therefore a constant. The constant is found to be zero by putting $u = 0$, so that the two are equal. By using this result

$$H'(u) = \frac{b}{2} \left[\frac{1}{r-a} + \frac{1}{r+a} \right].$$

A comparison of the principal parts at poles and of expansions at $u = 0$ as before, shows that

$$\frac{1}{b} H'(u) = -R_\mu[\wp(u - \omega_\mu) + e_\lambda] + R_\nu[\wp(u - \omega_\nu) + e_\lambda].$$

If r and G are expanded at $u = 0$ by Taylor's formula with the help of (9) and the equation $G' = ba/r$ respectively, and the derivatives r_a and G_a then taken, it is found that $H(0) = 0$. The equation for $H'(u)$ therefore gives, when integrated,

$$\begin{aligned} \frac{1}{b} H(u) &= R_\mu [\zeta(u - \omega_\mu) + \eta_\mu + e_\lambda u] \\ &\quad - R_\nu [\zeta(u - \omega_\nu) + \eta_\nu + e_\lambda u]. \end{aligned}$$

* Schwarz, *l. c.*, p. 12.

From this result the expression for Ψ_a is

$$\Psi_a = \frac{1}{2} H(2\omega_1) = b(R_\mu - R_\nu)(\eta_1 + e_\lambda \omega_1),$$

since $\zeta(u)$ has the property*

$$\zeta(u + 2\omega_\rho) = \zeta(u) + 2\eta_\rho, \quad \rho = \lambda, \mu, \nu.$$

The sign of Ψ_a can be determined. From (16)

$$R_\mu - R_\nu > 0 \text{ for Case } A; \quad R_\mu - R_\nu < 0 \text{ for Case } B.$$

In Case A^*

$$\eta_1 + e_1\omega_1 = \frac{\pi^2}{2\omega_1} \left\{ \frac{1}{2} + \sum_n \frac{4h^{2n}}{(1+h^{2n})^2} \right\} > 0,$$

where $h = e^{\pi i \omega_3 / \omega_1}$ is a real positive quantity; and in Case B ,

$$\eta_1 + e_2\omega_1 = \frac{\pi^2}{2\omega_1} \sum_n \frac{4h^{2n-1}}{(1+h^{2n-1})^2} > 0.$$

Ψ therefore increases as a increases from 0 to $a - b$, and decreases as a increases from $a - b$ to $a + b$.

The limits of Ψ in Case A are 0 and ∞ , and in Case B , ∞ and $\frac{\pi}{2} \sqrt{\frac{b}{a+b}}$.

For in either case†

$$\omega_1 = \frac{1}{\sqrt{e_1 - e_3}} \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-\kappa^2 t^2)}}, \quad \kappa^2 = \frac{e_2 - e_3}{e_1 - e_3}.$$

From equations (8) the limits of $e_1 - e_3$ and κ can be found. Then

$$\lim_{a=0} \omega_1 = \frac{2}{\sqrt{a^2 - b^2}} \int_0^1 \frac{dt}{\sqrt{1-t^2}} = \frac{\pi}{\sqrt{a^2 - b^2}},$$

and similarly

$$\lim_{a=a-b} \omega_1 = \infty, \quad \lim_{a=a+b} \omega_1 = \frac{\pi}{2\sqrt{b(a+b)}}.$$

The integral in (11) shows that

$$\lim_{a=0} \Psi = 0, \quad \lim_{a=a-b} \Psi = \infty, \quad \lim_{a=a+b} \Psi = \frac{\pi}{2} \sqrt{\frac{b}{a+b}}.$$

For Case A and a near to zero the geodesic line is therefore only a little

* Schwarz, *l. c.*, p. 9.

† Schwarz, *l. c.*, pp. 36 and 32.

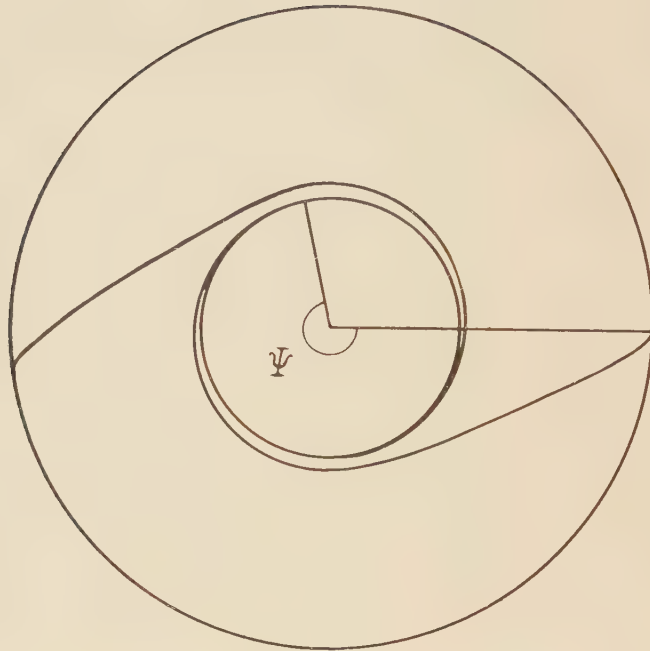


FIG. 6.

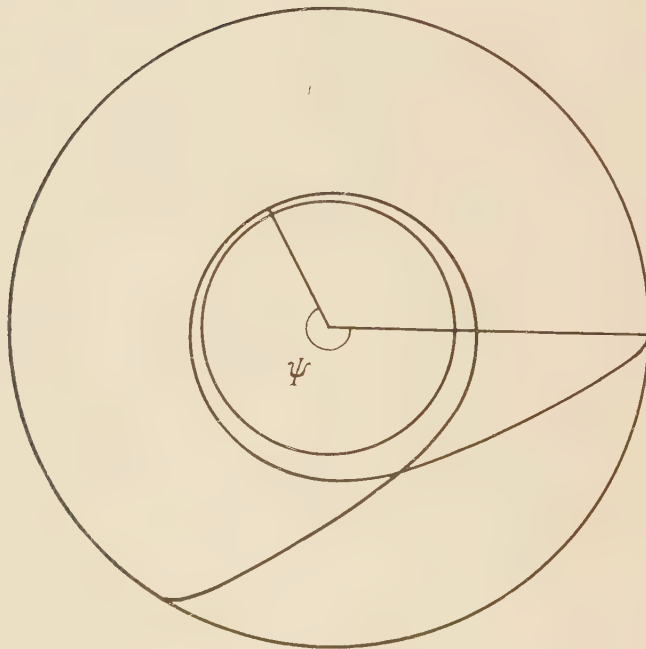


FIG. 7.

different from a meridian. As a increases it takes first the form of figure 3, then that of figure 6, and finally if a is near to $a - b$ winds many times around the inner equator before crossing it. For Case B and a near to $a - b$, the geodesic line winds many times around the inner equator before touching the circle $r = a$. As a increases it takes successively the forms shown in figures 4 and 7. When a is near to $a + b$ the line almost coincides with the outer equator, and intersects it in two points whose radii make an angle a little greater than $\pi\sqrt{\frac{b}{a+b}}$ with each other. The line of figure 5 for $a = a - b$ is a limiting line between those of Cases A and B .

5. Determination of Conjugate Points. The conjugates to a point $a(u = u_0)$ upon a geodesic line are determined by the zeros of $\Theta(u, u_0)$ which lie nearest to u_0 . The functions $\theta_1(u)$ and $\theta_2(u)$ of §1 are, from (11),

$$(19) \quad \begin{cases} \theta_1(u) = \frac{r_u}{b \sin \phi} \left\{ H(u) - G_a(\beta) \right\}, \\ \theta_2(u) = -\frac{a}{r(\beta)} \frac{r_u}{\sin \phi}. \end{cases}$$

Since from (9)

$$r_u = b \sin \phi \sqrt{r^2 - a^2},$$

the value of $\Theta(u, u_0)$ is

$$(20) \quad \Theta(u, u_0) = C_3 \sqrt{r^2 - a^2} [H(u) - H(u_0)],$$

C_3 being a constant. The real zeros of this function are to be investigated.

For a point on a line of Case A , $\Theta(u, u_0)$ has only the zero $u = u_0$. The factor $\sqrt{r^2 - a^2}$ is never zero, for r is always greater than a . $H(u) - H(u_0)$ vanishes at $u = u_0$, is finite for all real values of u , and has a positive derivative. *Therefore no point on a geodesic line of Case A can have a conjugate.*

When the geodesic line is one of case B and tangent to the circle $r = a$ at the point a , $r(u_0)$ has the value a , and $\left[\frac{dr}{du}\right]_{u=u_0}$ therefore vanishes. In this special case the expression for $\Theta(u, u_0)$, when calculated from the functions (19), is simply $C_4 \sqrt{r^2 - a^2}$, where C_4 is a constant. *The conjugate points to a point of tangency of a geodesic line of Case B with the circle $r = a$, $z > 0$, are therefore the two nearest points of tangency with the circle $r = a$, $z < 0$.*

For other values of u_0 in Case B , $\Theta(u, u_0)$ in (20) has no zeros due to the

factor $\sqrt{r^2 - a^2}$. For, from the properties of the $\wp$ -function in (7), it follows that the only real values of u which make $r = a$ are $u = (2k + 1)\omega_1$, where k is any integer. At such points $\sqrt{r^2 - a^2}$ has a zero of order one and $H(u)$ has a pole of order one, so that $\Theta(u, u_0)$ is finite and does not vanish. The factor $H(u) - H(u_0)$ has a zero in every interval from $(2k - 1)\omega_1$ to $(2k + 1)\omega_1$. For at either of these values $H(u)$ is infinite, and at other points its derivative is positive. In any such interval therefore $H(u)$ increases from $-\infty$ to $+\infty$ and takes the value $H(u_0)$ once. The geodesic line can be divided into sections between consecutive points of tangency with the circles $r = a, z > 0$ and $r = a, z < 0$. *Any point on a certain section of a line of Case B has two conjugates, one in each adjacent section.*

As u_0 traverses its interval $(2k - 1)\omega_1$ to $(2k + 1)\omega_1$, the value u' making

$$(21) \quad H(u') = H(u_0)$$

traverses its interval. For if u' is regarded as a function of u satisfying (21), its derivative is positive, and in order to make $H(u')$ take all possible values $H(u_0)$, u' must traverse its whole interval. Therefore *if the point a moves from one end of a section of the geodesic line to the other, its conjugates traverse the two adjacent sections in the same direction.*

It is interesting to notice that the radii of a and its conjugate a' make an angle with each other which is never less than 2Ψ . This follows from the property,

$$H(u + 2\omega_1) = H(u) + H(2\omega_1),$$

where $H(2\omega_1) < 0$ as in §4. The value of $H(u_0 + 2\omega_1)$ is less than $H(u_0)$, so that u' must lie beyond $u_0 + 2\omega_1$ in order to satisfy (21). As an illustration, a point upon the outer equator will have a conjugate beyond the next intersection of the geodesic line with that equator, as indicated in figure 4. The conjugate point marks the place where the geodesic line ceases to be a relative minimum. There may be a curve on the surface joining a with some point b between a and a' , and which is shorter than the corresponding arc of the geodesic line, but a neighborhood of ab , perhaps necessarily small, can always be found in which no such curve exists.

The conjugates of a point a upon either of the two equators, or a meridian, or the line (6) for $a = a - b$, cannot be conveniently determined from

$\Theta(u, u_0)$ as expressed in (20). In these cases the elliptic functions degenerate. But the conjugates can be determined by means of the equation*

$$(22) \quad \frac{d^2 m}{ds^2} + km = 0,$$

where k is the curvature of the surface and s the arc of the geodesic line measured from a . If m is an integral of (22) which vanishes for $s = 0$, the conjugates to a are determined by the values of s nearest to zero which make m vanish. The equation (22) for geodesic lines is the transform of one which has $\Theta(u, u_0)$ as an integral in the problem of the Calculus of Variations. m differs from $\Theta(u, u_0)$ only by a factor which never becomes zero.

Upon the anchor ring

$$k = \frac{r-a}{br}.$$

From (2) and (5),

$$\frac{ds}{dr} = \frac{ds}{d\phi} \frac{d\phi}{dr} = -\frac{br}{\sqrt{R(r)}}.$$

Equation (22) has therefore the integral

$$m = \sqrt{r^2 - a^2} \int_0^s \frac{ds}{r^2 - a^2},$$

which is indeed $\Theta(u, u_0)$ in different form and multiplied by a constant. When a is 0 or $a = b$, m has only the zero $s = 0$. Therefore a point on a meridian circle, or on the curve in figure 5, can have no conjugate.

For the inner and outer equators k has the values,

$$k = \mp \frac{1}{b(a \mp b)},$$

with the upper and lower signs respectively. Equation (22) shows that for the inner equator m , $\frac{dm}{ds}$, and $\frac{d^2 m}{ds^2}$ have the same sign, so that if $m = 0$ for $s = 0$, it cannot vanish for any other value of s . For the outer equator, (22) has the integral

$$m = \sin \frac{s}{\sqrt{b(a+b)}} = \sin \psi \sqrt{\frac{a+b}{b}},$$

* Darboux, *Leçons sur la théorie générale des surfaces*, vol. 3, p. 97.

and the first values of ψ for which this vanishes are

$$\psi = \pm \pi \sqrt{\frac{b}{a+b}}.$$

These results show that *no point on the inner equator has a conjugate. But every point on the outer equator has two conjugates whose radii make the angles $\pm \pi \sqrt{\frac{b}{a+b}}$ with the radius of the given point.* The latter result might have been deduced in §4 from the fact that the intersections of the geodesic lines through a with the outer equator, have a limiting point whose radius makes the angle $\pi \sqrt{\frac{b}{a+b}}$ with that of a .

From the foregoing paragraphs it follows that *a geodesic line on the anchor ring which has points in common with the inner equator or approaches it asymptotically, is a relative minimum between any two of its points. But any point on a geodesic line which does not touch the inner equator has two conjugates.*

6. Classification of the Points upon the Anchor Ring. A point can be classified according to Mangoldt's scheme by finding out all of the geodesic lines which pass through it. If the given point has a conjugate on any one of them, it cannot be of the first kind. From the results of §5, any point on the inner equator is of the first kind because the geodesic lines through it can have no conjugates.

Through any other point (r_0, ψ_0, z_0) of the anchor ring there are two geodesic lines corresponding to each value of a less than r_0 , and symmetrical with respect to the plane of the meridian $\psi = \psi_0$. This admits of an analytical proof, but is evident geometrically. For any two lines of Case *A* (or for $a = a - b$, see figure 5) which have the same value of a but wind in opposite directions about the ring, cut the circle $r = r_0, z = z_0$ at angles χ and $-\chi$ respectively; and two such intersections can be made to coincide with the given point by rotations about the z -axis. Similarly any line of Case *B* which has $a < r_0$ cuts the circle $r = r_0, z = z_0$ at two different angles, and by rotations can be made to pass through the given point in two directions symmetrical with respect to the meridian $\psi = \psi_0$. The two lines passing through the given point and corresponding to $a = a - b$ are limiting lines between those of Cases *A* and *B* (see figure 8). From (4) there are no lines through the given point

for $a > r_0$. Since the lines of Case *B* and the outer equator are the only ones which have conjugate points, the following theorem can be stated :

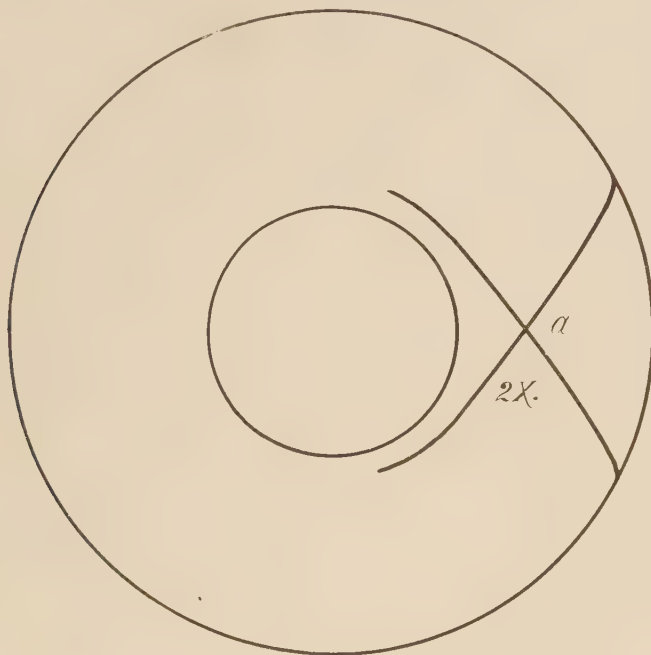


FIG. 8.

The points on the inner equator of the anchor ring are of the first kind according to Mangoldt's classification, and all others are of the second kind. The only geodesic lines on which there are conjugates to a point (r_0, ψ_0, z_0) of the second kind are those which pass through that point and make an angle less than χ_0 with the parallel circle $r = r_0, z = z_0$ where

$$\cos \chi_0 = \frac{a - b}{r_0} .$$

AUTOBIOGRAPHY.

I was born in Chicago, on the ninth day of May, 1876. My earlier education was received in the Chicago public schools. In 1893 I entered The University of Chicago, and in 1897 received the degree of Bachelor of Science. After one year of graduate study in mathematics and astronomy, I received the degree of Master of Science. I continued my studies in the graduate school for two years, in mathematics under Professors Moore, Bolza, and Maschke, in astronomy under Doctors Laves and Moulton, and in physics under Professor Michelson. I feel deeply indebted to these instructors for their kindness and encouragement during my work as a graduate student.

GILBERT AMES BLISS.

